

FIBERS OF L^∞ ALGEBRA.

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ABSTRACT. It is shown that Gelfand transforms of elements $f \in L^\infty(\mu)$ are constant at almost every fiber $\Pi^{-1}(\{x\})$ of the spectrum of $L^\infty(\mu)$ in the following sense: for each $f \in L^\infty(\mu)$ there is an open dense subset $U = U(f)$ of this spectrum having full measure and such that the Gelfand transform of f is constant on the intersection $\Pi^{-1}(\{x\}) \cap U$.

The proof of the main result bases on topological and measure properties of the spectrum of $L^\infty(\mu)$, (see [1], [2] I.9). This result is related to certain techniques connected with studying abstract approach to A-measures problem and corona problem.

In this note we consider $C(X)$, the algebra of all complex-valued continuous functions on a compact space X . Moreover we assume

- (*) μ is a regular Borel probabilistic measure on X such that X is equal to the closed support of μ

The set $L^\infty(\mu)$ of equivalence classes $[f]$ of $[\mu]$ essentially bounded measurable functions f on X is a commutative C^* -algebra under standard operations.

Let Y be the spectrum of $L^\infty(\mu)$. By Gelfand-Naimark theorem, $L^\infty(\mu)$ is isometrically isomorphic (by the Gelfand transform $[f] \rightarrow \widehat{[f]}$) to $C(Y)$.

Let $y \in Y$ and define a functional Π_y on $C(X)$ as follows:

$$(1) \quad \Pi_y(f) := \widehat{[f]}(y) \quad \text{for } f \in C(X).$$

Since

$$\Pi_y(fg) = \widehat{[fg]}(y) = \widehat{[f][g]}(y) = (\widehat{[f]}\widehat{[g]})(y) = \widehat{[f]}(y)\widehat{[g]}(y) = \Pi_y(f)\Pi_y(g),$$

we conclude that Π_y is a linear-multiplicative functional on $C(X)$, so it can be identified with some point in X . Using this identification we can write $f(\Pi_y) = \Pi_y(f)$ for $f \in C(X)$. Consider the mapping $\Pi : y \rightarrow \Pi_y$ and observe that $f \circ \Pi = \widehat{[f]}$ for $f \in C(X)$. Hence $f \circ \Pi$ is a continuous function on Y for each $f \in C(X)$. Consequently, since Gelfand topologies on X and Y are equal to the restrictions of the weak-star topologies to X and Y respectively, we have the following

Lemma 1. *Projection $\Pi : Y \rightarrow X$ is continuous.*

Let us consider the sequence of mappings

$$(2) \quad C(X) \ni f \rightarrow [f] \rightarrow \widehat{[f]} \in C(Y).$$

2010 *Mathematics Subject Classification.* Primary: 46J10; Secondary: 46E30, 28A20.

Key words and phrases. function algebra, measure, L^∞ algebra, fiber.

By the assumption (*), the first mapping is an isometry into $L^\infty(\mu)$. The last one is the Gelfand transform: $L^\infty(\mu) \rightarrow C(Y)$ which is also an isometry. Hence, by (1) we have for $f \in C(X)$

$$(3) \quad \sup_{x \in X} |f(x)| = \|f\| = \|\widehat{[f]}\| = \sup_{x \in \Pi(Y)} |f(x)|.$$

By Lemma 1, the set $\Pi(Y)$ is compact, and hence closed in X which by (3) implies that $\Pi(Y)$ contains Shilov boundary of $C(X)$. Consequently $\Pi(Y)$ must be equal to X . So we have

Proposition 2.

- (1) *Up to the isometric equivalence given by (2), $C(X)$ can be considered as a closed subalgebra of $L^\infty(\mu)$.*
- (2) *Each element $x \in X$ as a linear-multiplicative functional on $C(X)$ has a linear-multiplicative extension $y : [f] \rightarrow \widehat{[f]}(y)$ to the whole $L^\infty(\mu)$.*

From now on we will not distinguish in writing Borel, $[\mu]$ essentially bounded functions on X from their equivalence classes in $L^\infty(\mu)$. By the above consideration, if $f \in C(X)$ then \widehat{f} is constant on each fiber $\Pi^{-1}(\{x\})$ for $x \in X$.

Since we identify $L^\infty(\mu)$ with $C(Y)$, Riesz Representation Theorem gives a regular positive Borel measure $\tilde{\mu}$ on Y "representing μ " i.e. such that $\|\tilde{\mu}\| = \|\mu\|$ and

$$(4) \quad \int f d\mu = \int \widehat{f} d\tilde{\mu} \quad \text{for } f \in L^\infty(\mu).$$

For any Borel $E \subset X$ its characteristic function $\widehat{\chi_E}$ as an idempotent in $C(Y)$ is of the form χ_{U_E} , thus assigning a closed-open set U_E in Y to any measurable $E \subset X$. Applying (4) to χ_E we get for any Borel subset E of X the equality

$$(5) \quad \mu(E) = \tilde{\mu}(U_E).$$

Moreover (Lemma 9.1 and Corollary 9.2 of [2]) we have

Lemma 3. *The family $\{U_E : E \subset Y, E \text{ measurable}\}$ form a basis for the topology of Y . If U is an open non-empty subset of Y , then $\tilde{\mu}(U) > 0$.*

Lemma 4. *If E, F are Borel subsets of X , and $E \subset F$ then $\widehat{\chi_E} \leq \widehat{\chi_F}$ and $U_E \subset U_F$.*

Proof. If $E \subset F$ then $\chi_E = \chi_E \cdot \chi_F$. Hence $\widehat{\chi_E} = \widehat{\chi_E} \cdot \widehat{\chi_F}$ which means that $\widehat{\chi_E} \leq \widehat{\chi_F}$. Since $\chi_{U_E} = \widehat{\chi_E}$ and $\chi_{U_F} = \widehat{\chi_F}$, we have $U_E \subset U_F$. \square

Lemma 5. *If $E \subset X$ is open then $\Pi^{-1}(E) \subset U_E$ and $\chi_{\Pi^{-1}(E)} \leq \widehat{\chi_E}$. If $E \subset X$ is closed then $\Pi^{-1}(E) \supset U_E$ and $\chi_{\Pi^{-1}(E)} \geq \widehat{\chi_E}$.*

Proof. Let E be open in X and $x \in E$. Then there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 1$ and $f \leq \chi_E$. Hence \widehat{f} is equal 1 on $\Pi^{-1}(\{x\})$ and $f = f \cdot \chi_E$, which implies $\widehat{f} = \widehat{f} \cdot \widehat{\chi_E} = \widehat{f} \cdot \chi_{U_E}$. Consequently $\widehat{\chi_{U_E}}$ is equal 1 on $\Pi^{-1}(\{x\})$ which means that $\Pi^{-1}(\{x\}) \subset U_E$. Since x was an arbitrary point of E , we have $\Pi^{-1}(E) \subset U_E$. Then also $\chi_{\Pi^{-1}(E)} \leq \chi_{U_E} = \widehat{\chi_E}$.

If E is closed then $X \setminus E$ is open and $\chi_E \cdot \chi_{X \setminus E} = 0$, $\chi_E + \chi_{X \setminus E} = 1$. Consequently $\chi_{U_E} \chi_{U_{X \setminus E}} = \widehat{\chi_E} \cdot \widehat{\chi_{X \setminus E}} = 0$ and $\chi_{U_E} + \chi_{U_{X \setminus E}} = \widehat{\chi_E} + \widehat{\chi_{X \setminus E}} = 1$. It means that

$U_E \cap U_{X \setminus E} = \emptyset$ and $U_E \cup U_{X \setminus E} = Y$ which implies the desired statement for closed sets. \square

Remark 6. Till now the regularity of μ has not been used.

Lemma 7. *If E is a Borel subset of X then*

$$(6) \quad \mu(E) = \tilde{\mu}(\Pi^{-1}(E)) = \tilde{\mu}(U_E).$$

If $E \subset X$ is open then $\overline{\Pi^{-1}(E)} = U_E$. If $E \subset X$ is closed then $\text{int}(\Pi^{-1}(E)) = U_E$.

Proof. By the regularity of μ , for any $\varepsilon > 0$ we can find a compact set $K \subset X$ and an open set $V \subset X$ such that $K \subset E \subset V$ and $\mu(V \setminus K) < \varepsilon$. Also there exists $f \in C(X)$ such that $\chi_K \leq f \leq \chi_V$. By the continuity of f we have $\chi_{\Pi^{-1}(K)} \leq \hat{f} \leq \chi_{\Pi^{-1}(V)}$. (Proposition 2 and the consideration following it). Hence $|\mu(E) - \int f d\mu| < \varepsilon$ and $|\tilde{\mu}(\Pi^{-1}(E)) - \int \hat{f} d\tilde{\mu}| < \varepsilon$ which by (4) and free choice of ε gives $\mu(E) = \tilde{\mu}(\Pi^{-1}(E))$. The second equality in (6) we get by (5).

If E is closed then $U_E \subset \Pi^{-1}(E)$ by Lemma 5. So $U_E \subset \text{int}(\Pi^{-1}(E))$ and $\text{int}(\Pi^{-1}(E)) \setminus U_E$ is open since U_E is closed-open. Consequently $\text{int}(\Pi^{-1}(E)) = U_E$ by Lemma 3.

The assertion for open sets follows from the equalities $\text{int}(\Pi^{-1}(E)) = Y \setminus \overline{\Pi^{-1}(X \setminus E)}$ and $U_E = Y \setminus U_{X \setminus E}$. \square

Theorem. *If μ is a measure satisfying $(*)$, Y is the spectrum of $L^\infty(\mu)$, and $h \in L^\infty(\mu)$, then there exists an open dense subset U of Y with $\tilde{\mu}(U) = \tilde{\mu}(Y)$ such that \hat{h} is constant on $\Pi^{-1}(\{x\}) \cap U$ for all $x \in X$.*

Proof. Let $h \in L^\infty(\mu)$, and let $\varepsilon > 0$. By Lusin Theorem there is $g \in C(X)$ with $\|g\| \leq \|h\|$ and a closed set $Z \subset X$ such that $\mu(X \setminus Z) < \varepsilon$ while $Z \subset \{g = h\}$. By Lemma 7 we have

$$U_Z = \text{int}(\Pi^{-1}(Z)), \quad \tilde{\mu}(U_Z) = \mu(Z) > 1 - \varepsilon.$$

Since $Z \subset \{g = h\}$ then $\chi_Z \cdot (g - h) = 0$. Consequently $\chi_{U_Z} \cdot (\hat{g} - \hat{h}) = \widehat{\chi_Z} \cdot (\hat{g} - \hat{h}) = 0$ which implies

$$\{\hat{g} \neq \hat{h}\} \cap U_Z = \emptyset.$$

Put $Z_1 := Z$ and $\varepsilon = 1/2$. Repeating the previous construction we find a sequence $\{g_n\} \subset C(X)$ and a sequence $\{Z_n\}$ of closed subsets of X such that $Z_n \subset \{g_n = h\}$ and $\mu(X \setminus Z_n) < 1/2^n$. Then

$$\tilde{\mu}(U_{Z_n}) = \mu(Z_n) > 1 - 1/2^n, \quad \{\hat{g}_n \neq \hat{h}\} \cap U_{Z_n} = \emptyset.$$

The last equality implies that \hat{h} is constant on each $\Pi^{-1}(\{x\}) \cap U_{Z_n}$ for all $x \in X$ and $n \in \mathbb{N}$. We define a sequence of open sets as follows:

$$U_1 := U_{Z_1}, \quad U_n := U_{Z_n} \setminus \Pi^{-1}(Z_1 \cup \dots \cup Z_{n-1}).$$

By the above definition and Lemma 5, for $k \in \mathbb{N}$ we have $\Pi^{-1}(Z_k) \supset U_{Z_k} \supset U_k$, hence $Z_k \supset \Pi(U_{Z_k}) \supset \Pi(U_k)$, and consequently

$$(7) \quad \Pi(U_n) \cap \Pi(U_m) = \emptyset \quad \text{for } n \neq m$$

since $\Pi(U_n) \cap Z_k = \emptyset$ for $k < n$. By Lemma 7 we have $\tilde{\mu}(\Pi^{-1}(Z_n) \setminus U_{Z_n}) = 0$ and hence

$$(8) \quad \begin{aligned} \tilde{\mu}(U_n) &= \tilde{\mu}(U_{Z_n} \setminus \Pi^{-1}(Z_1 \cup \dots \cup Z_{n-1})) = \tilde{\mu}(\Pi^{-1}(Z_n) \setminus \Pi^{-1}(Z_1 \cup \dots \cup Z_{n-1})) \\ &= \tilde{\mu}(\Pi^{-1}(Z_n \setminus (Z_1 \cup \dots \cup Z_{n-1}))) = \mu(Z_n \setminus (Z_1 \cup \dots \cup Z_{n-1})). \end{aligned}$$

Put now $Z'_1 := Z_1$ and $Z'_n := Z_n \setminus (Z_1 \cup \dots \cup Z_{n-1})$ for $n > 1$. All the sets $\{Z'_n\}$ are pairwise disjoint and a direct calculation gives the equality $Z'_n \cup Z'_{n-1} \supset Z_n \setminus (Z_1 \cup \dots \cup Z_{n-2})$ which by induction leads to the assertion $Z'_1 \cup \dots \cup Z'_n \supset Z_n$. Hence, by (8) and pairwise disjointness of $\{U_n\}$ and $\{Z'_n\}$, we get

$$\begin{aligned} \tilde{\mu}(U_1 \cup \dots \cup U_n) &= \tilde{\mu}(U_1) + \dots + \tilde{\mu}(U_n) = \mu(Z'_1) + \dots + \mu(Z'_n) \\ &= \mu(Z'_1 \cup \dots \cup Z'_n) \geq \mu(Z_n) > 1 - 1/2^n. \end{aligned}$$

Put $U := \sum_{n=1}^{\infty} U_n$. Hence U is open, $\tilde{\mu}(U) = 1 = \tilde{\mu}(Y)$, and consequently, by Lemma 3, U is dense in Y . The function \hat{h} is constant on each $\Pi^{-1}(\{x\}) \cap U_n$ for all $x \in X$ and $n \in \mathbb{N}$ and sets $\Pi(U_n)$, $n \in \mathbb{N}$ are pairwise disjoint by (7). It means that each fiber $\Pi^{-1}(\{x\})$ intersects at most one of the sets U_n . Hence \hat{h} is constant on each $\Pi^{-1}(\{x\}) \cap U$ for all $x \in X$. □

Remark 8. If the closed support of μ is not equal to X then $L^\infty(\mu)$ is isometrically isomorphic to the algebra $\{f|_{\text{supp}(\mu)} : f \in L^\infty(\mu)\}$. In such a case $\Pi^{-1}(\{x\}) = \emptyset$ for all x outside of the closed support of μ . Assuming that each function is constant on empty set we conclude that the result of Theorem holds true also when the closed support of μ is a proper subset of X .

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